Solvability of general backward stochastic Volterra integral equations with non-Lipschitz conditions *

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Abstract

In this paper we study the unique solvability of backward stochastic Volterra integral equations (BSVIEs in short), in terms of both the M-solutions introduced in [17] and the adapted solutions in [6], [12] or [14]. A general existence and uniqueness of M-solutions is proved under non-Lipschitz conditions by virtue of a briefer argument than the one in [17], which extends the results in [17]. For the adapted solutions, the unique solvability of BSVIEs under more general stochastic non-Lipschitz conditions is obtained, which generalize the results in [6], [12] and [14].

Keywords: Backward stochastic Volterra integral equations, Adapted M-solutions, Non-Lipschitz condition, stochastic Lipschitz coefficients, adapted solutions

1 Introduction

Let $\{W_t\}_{t\in[0,T]}$ be a d-dimensional Wiener process defined on a probability space $(\Omega,\mathcal{F},P)$ and $\{\mathcal{F}_t\}_{t\in[0,T]}$ denote the natural filtration of $\{W_t\}$, such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. This paper is motivated by the recent work of Yong ([15], [17]), which studied an extension of backward stochastic differential equations (BSDEs in short), i.e. backward stochastic Volterra integral equations (BSVIEs in short). The nonlinear BSDEs of the form

$$Y(t)=\xi+\int_t^T g(s,Y(s),Z(s))ds-\int_t^T Z(s)dW(s), \tag{1}$$

initiated by Pardoux and Peng [11], have been studied extensively in the past two decades. We refer the reader to the books of Ma and Yong [7], Yong and Zhou [18] and

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the survey paper of El Karoui, Peng and Quenez [5] for the detailed accounts of both theory and application (especially in mathematical finance and stochastic control) for such equations. On the other hand, BSVIEs of the form

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \tag{2}$$

were firstly introduced by Yong [15]. We refer the reader to [15], [16] and [17] for both theory and application in dynamic risk measure and optimal control. As to the adapted solution of BSVIE (2) (g is independent of $Z(s,t)$ or $\psi(t) = \xi$), see [1], [6], [12], [14], and the references cited therein.

No matter the M-solution in [15], [17] and [16], or the adapted solution in [6], [12] and [14], they made at least one of the following assumptions, 1) g is independent of $Z(s,t)$, 2) the terminal condition is $\mathcal{F}_T$-measurable random variable ξ, 3) the Lipschitz condition, moreover, the coefficient is deterministic, 4) the deterministic non-Lipschitz condition. In this paper, we consider the general case of the above two kinds of solutions respectively. At first we will prove the unique solvability of M-solutions with a new method. The reason is at least two-fold. On the one hand, before proving the unique existence of M-solution, we should make many preparations if we use the method in [17], such as the solvability of solutions of certain stochastic Fredholm integral equation and some other estimates of M-solution for certain simple BSVIEs. On the other hand, BSVIEs do not have time-consistency (or semigroup) property, and the process Z has two parameters, so if we use induction method in the solution procedure for BSVIEs, we have to use four steps as in [17], which seems rather complicated and sophisticated. So we will introduce a new convenient method from other perspective, that is, when T is finite, we can use an equivalent norm in $\mathcal{H}^2[0,T]$ as follows:

$$\|(y(\cdot), z(\cdot,\cdot))\|_{\mathcal{H}^2[0,T]} = \left[E\int_0^T e^{\beta t}|y(t)|^2dt + E\int_0^T\int_0^T e^{\beta s}|z(t,s)|^2dsdt\right]^{\frac{1}{2}},$$

where β is a positive constant, and then we will prove the results of M-solutions within the new norm with one step. This is our first contribution in this paper. When g is independent of $Z(s,t)$, we can use the estimate in lemma 3.1 and the method in [14] to get the solvability of M-solution of BSVIE (2) (g is independent of $Z(s,t)$) under non-Lipschitz condition. However, as to the general form of BSVIE (2), the method in [14] does not work any more because of the appearance of $E\int_t^T |Z(s,t)|^2ds$, which can be estimated by means of Malliavin calculus, see [17], and this will make the problem more complicated. So we have to replace $e^{\beta t}E|Y(t)|^2 + E\int_t^T e^{\beta s}|Z(t,s)|^2ds$, as in [14], with a weaker form $\int_u^T e^{\beta t}E|Y(t)|^2dt + E\int_u^T\int_t^T e^{\beta s}|Z(t,s)|^2dsdt$, $u \in [0,T]$. We also have to prove a new lemma by means of the definition of concave function, then we can obtain the unique existence of M-solution of BSVIE (2) under non-Lipschitz condition, which generalize the result in [15], [17] and [16]. Thirdly, we claim that It$\hat{o}$ formula plays a key role in the BSDEs case, as well as the BSVIEs case in [14]. One question is can we get the solvability of adapted solution of (2) (g is independent of $Z(s,t)$)

under stochastic non-Lipschitz conditions without involving Itô formula? The answer is positive and we will prove it in the following, which generalize the result in [6] and [14].

Recently the author considered the unique solvability of M-solution under non-Lipschitz condition by induction in [12]. Note that our method here is different from it, moreover, briefer than it. On the other hand, the assumption on the coefficients in [12] is also much stronger than ours here.

The paper is organized as follows. In Section 2, we give some preliminary results and notations which are needed in the following sections. An important estimate for M-solutions (or adapted solutions) is presented in Subsection 3.1. With this estimate, we give the existence and uniqueness result of M-solutions under Lipschitz condition in Subsection 3.2. The case of adapted solutions is also treated. In Subsection 3.3, we consider the unique solvability of M-solutions (adapted solutions respectively) under non-Lipschitz case. At last examples of M-solutions and adapted solutions under non-Lipschitz condition is present.

2 Preliminaries

In this section, we will make some preliminaries. $\forall R, S \in [0,T]$, in the following we denote $\Delta^c[R,S] = \{(t,s) \in [R,S]^2; t \leq s\}$, $\Delta^c = \Delta^c[0,T]$, $\Delta[R,S] = \{(t,s) \in [R,S]^2; t > s\}$, and $\Delta = \Delta[0,T]$. Let us first introduce some spaces. Let β be a positive constant. $A(t)$ is a non-negative $\mathcal{F}_t$-adapted increasing process. Let $L^{2,\beta}_{\mathcal{F}_T}[0,T]$ be the set of the $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable processes $X : [0,T] \times \Omega \to R^m$ such that $Ee^{\beta A(T)} \int_0^T |X(t)|^2 dt < \infty$. We denote

$$\begin{aligned}
\mathcal{H}^{2,\beta}[R,S] &= L^{2,\beta}_{\mathcal{F}}[R,S] \times L^{2,\beta}(R,S; L^2_{\mathcal{F}}[R,S]), \\
\mathcal{H}^{2,\beta}_t[R,S] &= L^{2,\beta}_{\mathcal{F}}[R,S] \times L^{2,\beta}(R,S; L^2_{\mathcal{F}}[t,S]).
\end{aligned}$$

Here $L^{2,\beta}_{\mathcal{F}}[R,S]$ is the set of all adapted processes $X : [R,S] \times \Omega \to R^m$ such that $E\int_R^S e^{\beta A(s)}|X(s)|^2 ds < \infty$. $L^{2,\beta}(R,S; L^2_{\mathcal{F}}[R,S])$ is the set of all processes $Z : [R,S]^2 \times \Omega \to R^{m\times d}$ such that for almost all $t \in [R,S]$, $Z(t,\cdot)$ is $\mathcal{F}$-progressively measurable satisfying $E\int_R^S \int_R^S e^{\beta A(s)}|Z(t,s)|^2 ds dt < \infty$. $L^{2,\beta}(R,S; L^2_{\mathcal{F}}[t,S])$ is the set of all processes $Z(t,s) : \Delta^c[R,S] \times \Omega \to R^{m\times d}$ such that for almost all $t \in [R,S]$, $Z(t,\cdot)$ is $\mathcal{F}$-progressively measurable satisfying $E\int_R^S \int_t^S e^{\beta A(s)}|Z(t,s)|^2 ds dt < \infty$. Let $L^2_{\mathcal{F}_T}[0,T]$ be the set of the $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$ processes $X : [0,T] \times \Omega \to R^m$ such that $E\int_0^T |X(t)|^2 dt < \infty$. We also denote

$$\begin{aligned}
\mathcal{H}^2[R,S] &= L^2_{\mathcal{F}}[R,S] \times L^2(R,S; L^2_{\mathcal{F}}[R,S]), \\
\mathcal{H}^2_t[R,S] &= L^2_{\mathcal{F}}[R,S] \times L^2(R,S; L^2_{\mathcal{F}}[t,S]).
\end{aligned}$$

Here $L^2_{\mathcal{F}}[R,S]$ is the set of all adapted processes $X:[R,S]\times\Omega\to R^m$ such that $E\int_R^S |X(s)|^2ds<\infty$. $L^2(R,S;L^2_{\mathcal{F}}[R,S])$ is the set of all processes $Z:[R,S]^2\times\Omega\to R^{m\times d}$ such that for almost all $t\in[R,S]$, $Z(t,\cdot)$ is $\mathcal{F}$-progressively measurable satisfying $E\int_R^S\int_R^S |Z(t,s)|^2dsdt<\infty$. $L^2(R,S;L^2_{\mathcal{F}}[t,S])$ is the set of all processes $Z(t,s):\Delta^c[R,S]\times\Omega\to R^{m\times d}$ such that for almost all $t\in[R,S]$, $Z(t,\cdot)$ is $\mathcal{F}$-progressively measurable satisfying $E\int_R^S\int_t^S |Z(t,s)|^2dsdt<\infty$. Now we give two definitions needed in the sequel.

Definition 2.1 *Let $S\in[0,T]$. A pair of $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{H}^{2,\beta}[S,T]$ is called an adapted M-solution of BSVIE (2) on $[S,T]$ if (2) holds in the usual Itô's sense for almost all $t\in[S,T]$ and, in addition, the following holds:*

$$Y(t)=E^{\mathcal{F}_S}Y(t)+\int_S^t Z(t,s)dW(s),\quad t\in[S,T].$$

Definition 2.2 *A pair of $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{H}^{2,\beta}_t[0,T]$ is called an adapted solution of the following simple BSVIE (3) if (3) holds in the usual Itô's sense*

$$Y(t)=\psi(t)+\int_t^T g(t,s,Y(s),Z(t,s))ds-\int_t^T Z(t,s)dW(s),\quad t\in[0,T]. \tag{3}$$

In [17], the author gave the definition of M-solution of BSVIE in $\mathcal{H}^2[0,T]$. The author also considered the existence and uniqueness of adapted solution of (3) ($\psi(\cdot)$ is replaced with ξ) in $\mathcal{H}^2_t[0,T]$ in [14].

We give the following assumptions of g for BSVIE (2):

(H1) Let $g:\Delta^c\times R^m\times R^{m\times d}\times R^{m\times d}\times\Omega\to R^m$ be $\mathcal{B}(\Delta^c\times R^m\times R^{m\times d}\times R^{m\times d})\otimes\mathcal{F}_T$-measurable such that $s\to g(t,s,y,z,\zeta)$ is $\mathcal{F}$-progressively measurable for all $(t,y,z,\zeta)\in[0,T]\times R^m\times R^{m\times d}\times R^{m\times d}$, furthermore, g satisfies the Lipschitz conditions with stochastic coefficient, i.e., $\forall y,\ \overline{y}\in R^m,\ z,\ \overline{z},\ \zeta,\ \overline{\zeta}\in R^{m\times d}$,

$$\begin{aligned}&|g(t,s,y,z,\zeta)-g(t,s,\overline{y},\overline{z},\overline{\zeta})|\\ \le\ & L(t,s)(r_1(s)|y-\overline{y}|+r_2(s)|z-\overline{z}|+r_3(s)|\zeta-\overline{\zeta}|),\end{aligned}$$

where $(t,s)\in\Delta^c$, $r_1(s)$, $r_2(s)$ and $r_3(s)$ are non-negative adapted processes and we denote

$$\alpha^2(s)=r_1^2(s)+r_2^2(s)+r_3^2(s),\quad A(t)=\int_0^t\alpha^2(s)ds.$$

We assume that $\alpha^2(s)\ge\delta$, where δ is a positive constant, $\alpha(s)$ is a positive adapted process and $L(t,s)$ is a deterministic non-negative function. Furthermore, we assume

$$E\int_0^T\int_t^T e^{\beta A(s)}|g_0(t,s)|^2dsdt<\infty,$$

where $g_0(t,s)=g(t,s,0,0,0)$.

3 Main results for M-solutions

3.1 A basic estimate for M-solutions of BSVIEs

In this subsection, inspired by the method of estimating the adapted solutions of BSDEs in [4], we give a lemma which is needed in the following.

Lemma 3.1 *We consider the following simple BSVIE*

$$Y(t) = \psi(t) + \int_t^T f(t,s)ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \tag{4}$$

where $\psi(\cdot) \in L^{2,\beta}_{\mathcal{F}_T}[0,T]$, $f : \Delta^c \times \Omega \to R^m$ be $\mathcal{B}(\Delta^c) \otimes \mathcal{F}_T$-measurable such that $s \to f(t,s)$ is $\mathcal{F}$-progressively measurable for all $t \in [0,T]$, and $E\int_0^T \int_t^T e^{\beta A(s)}|f(t,s)|^2 dsdt < \infty$. Then (4) admits a unique adapted solution $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{H}^{2,\beta}_t[0,T]$, and we have the following estimate:

$$\begin{aligned}
& E\int_0^T e^{\beta A(s)}|Y(s)|^2 ds + E\int_0^T \int_t^T e^{\beta A(s)}|Z(t,s)|^2 dsdt \\
\leq\ & CE\int_0^T e^{\beta A(t)}|\psi(t)|^2 dt + CE\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2 dt \\
& +CE\int_0^T \int_t^T e^{\beta A(u)}|\psi(t)|^2 d\beta A(u)dt \\
& + CE\int_0^T \int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s)dt.
\end{aligned} \tag{5}$$

Furthermore,

$$\begin{aligned}
& E\int_0^T e^{\beta A(s)}|Y(s)|^2 ds + E\int_0^T \int_t^T e^{\beta A(s)}|Z(t,s)|^2 dsdt \\
\leq\ & CEe^{\beta A(T)}\int_0^T |\psi(t)|^2 dt + \frac{C}{\beta}E\int_0^T \int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt.
\end{aligned} \tag{6}$$

Hereafter C is a generic positive constant which may be different from line to line.

Proof. We consider a family of BSDEs with parameters t on $[0,T]$ in the following form:

$$\lambda(t,r) = \psi(t) + \int_r^T f(t,s)ds - \int_r^T \mu(t,s)dW(s), \quad t,r \in [0,T]. \tag{7}$$

By the classical existence and uniqueness theorem of BSDE in [11], there exists a unique solution $(\lambda(t,\cdot),\mu(t,\cdot))$ for every $t \in [0,T]$. Let $Y(t) = \lambda(t,t)$, $\quad Z(t,s) = \mu(t,s)$, $t \leq s$. Then we obtain the existence and uniqueness of the adapted solution for (4). From (7) we arrive at, $\forall r \geq t$,

$$\lambda(t,r) \quad = \quad E^{\mathcal{F}_r}\left(\psi(t) + \int_r^T f(t,s)ds\right),$$

and

$$\int_r^T Z(t,s)dW(s) \quad = \quad \int_r^T \mu(t,s)dW(s) = \psi(t) + \int_r^T f(t,s)ds - \lambda(t,r). \tag{8}$$

Especially when $r = t$,

$$\int_t^T Z(t,s)dW(s) = \int_t^T \mu(t,s)dW(s) = \psi(t) + \int_t^T f(t,s)ds - Y(t).$$

Now we estimate $E\int_0^T e^{\beta A(s)}|Y(s)|^2 ds + E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2 dsdt$. By Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}
\left|\int_s^T f(t,u)du\right|^2 \quad &= \quad \left|\int_s^T e^{\frac{-rA(u)}{2}}\alpha(u)e^{\frac{rA(u)}{2}}\frac{f(t,u)}{\alpha(u)}du\right|^2 \\
&\leq \quad \int_s^T e^{-rA(u)}\alpha^2(u)du \cdot \int_s^T e^{rA(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du \\
&\leq \quad \frac{1}{r}e^{-rA(s)}\int_s^T e^{rA(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du, \quad t,s\in[0,T],
\end{aligned} \tag{9}$$

where r is a positive constant. By taking $r = \frac{\beta}{2}$ in (9), we see that

$$\begin{aligned}
&\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s) \\
\leq \quad &\frac{4}{\beta}\int_t^T e^{\frac{\beta}{2}A(s)}\left(\int_s^T e^{\frac{\beta}{2}A(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du\right)d\frac{\beta}{2}A(s) \\
= \quad &\frac{4}{\beta}e^{\frac{\beta}{2}A(s)}\left(\int_s^T e^{\frac{\beta}{2}A(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du\right)\Bigg|_t^T \\
&+\frac{4}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}ds \\
\leq \quad &\frac{4}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}ds.
\end{aligned}$$

Therefore,

$$E\int_0^T\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s)dt \leq \frac{4}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt. \tag{10}$$

We also obtain the following result by taking $s = t$ and $r = \beta$ in (9),

$$E\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,u)du\right|^2 dt \leq \frac{1}{\beta}E\int_0^T\int_t^T e^{\beta A(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}dudt.$$

At first we estimate $E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt$. Obviously, we have $t,r\in[0,T]$,

$$\begin{aligned}
&\int_r^T e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)d\beta A(s)\\
= \ & e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)\Bigg|_r^T+\int_r^T e^{\beta A(s)}|Z(t,s)|^2ds. \qquad (11)
\end{aligned}$$

For arbitrary $t\in[0,T]$, we can rewrite (11) after taking $r=t$,

$$\begin{aligned}
&E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt\\
= \ & E\int_0^T\int_t^T e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)d\beta A(s)dt\\
&+E\int_0^T e^{\beta A(t)}\int_t^T |Z(t,u)|^2dudt. \qquad (12)
\end{aligned}$$

Now we give a estimate to the second expression in the right part of (12)

$$\begin{aligned}
&E\int_0^T e^{\beta A(t)}\int_t^T |Z(t,u)|^2dudt\\
= \ & E\int_0^T E\left(e^{\beta A(t)}\int_t^T |Z(t,u)|^2du\bigg|\mathcal{F}_t\right)dt\\
= \ & E\int_0^T e^{\beta A(t)}E\left(\left(\int_t^T Z(t,u)dW(u)\right)^2\Bigg|\mathcal{F}_t\right)dt\\
= \ & E\int_0^T e^{\beta A(t)}E\left(\left(\psi(t)+\int_t^T f(t,u)du-Y(t)\right)^2\Bigg|\mathcal{F}_t\right)dt\\
\leq \ & 3E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+3E\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,u)du\right|^2dt\\
&+3E\int_0^T e^{\beta A(t)}|Y(t)|^2dt\\
\leq \ & 3E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+\frac{3}{\beta}E\int_0^T\int_t^T e^{\beta A(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}dudt\\
&+3E\int_0^T e^{\beta A(t)}|Y(t)|^2dt. \qquad (13)
\end{aligned}$$

Obviously, we can use the similar method as (13) to estimate the first expression in the right part of (12) as follows:

$$\begin{aligned}
& E\int_0^T\int_t^T e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)d\beta A(s)dt \\
\leq\ & 3E\int_0^T\int_t^T e^{\beta A(s)}|\psi(t)|^2 d\beta A(s)dt \\
& +3E\int_0^T\int_t^T e^{\beta A(s)}\left(\int_s^T f(t,u)du\right)^2 d\beta A(s)dt \\
& +3E\int_0^T\int_t^T e^{\beta A(s)}|\lambda(t,s)|^2 d\beta A(s)dt. \qquad (14)
\end{aligned}$$

For the second expression in the right part of (14), inequality (10) implies that

$$\begin{aligned}
& 3E\int_0^T\int_t^T e^{\beta A(s)}\left(\int_s^T f(t,u)du\right)^2 d\beta A(s)dt \\
\leq\ & \frac{12}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt. \qquad (15)
\end{aligned}$$

It is time for us to estimate the third expression of (14). Observe that

$$\lambda(t,s) = E^{\mathcal{F}_s}\left(\psi(t)+\int_s^T f(t,u)du\right), \quad t\leq s,$$

so we deduce that,

$$|\lambda(t,s)|^2 \leq 2E(|\psi(t)|^2\,|\mathcal{F}_s) + 2E\left(\left|\int_s^T f(t,u)du\right|^2\middle|\,\mathcal{F}_s\right),$$

and

$$\begin{aligned}
& 3E\int_0^T\int_t^T e^{\beta A(s)}|\lambda(t,s)|^2 d\beta A(s)dt \\
\leq\ & 6E\int_0^T\int_t^T e^{\beta A(s)}|\psi(t)|^2 d\beta A(s)dt \\
& +6E\int_0^T\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s)dt \\
\leq\ & 6E\int_0^T |\psi(t)|^2dt\left(\int_t^T e^{\beta A(s)}d\beta A(s)\right) \\
& +\frac{24}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt. \qquad (16)
\end{aligned}$$

Equality (14), (15) and (16) implies that

$$\begin{aligned}
& E\int_0^T\int_t^T e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)d\beta A(s)dt \\
\leq\ & \frac{36}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt \\
& +9E\int_0^T|\psi(t)|^2dt\left(\int_t^T e^{\beta A(s)}d\beta A(s)\right).
\end{aligned} \tag{17}$$

From (12), (13) and (17), we also see that

$$\begin{aligned}
& E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt \\
\leq\ & 3E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+9E\int_0^T|\psi(t)|^2dt\left(\int_t^T e^{\beta A(s)}d\beta A(s)\right) \\
& +\frac{39}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt+3E\int_0^T e^{\beta A(t)}|Y(t)|^2dt.
\end{aligned}$$

Due to $Y(t)=E^{\mathcal{F}_t}\left(\psi(t)+\int_t^T f(t,s)ds\right)$, we have

$$\begin{aligned}
& E\int_0^T e^{\beta A(t)}|Y(t)|^2dt \\
\leq\ & 2E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+2E\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2dt \\
\leq\ & 2E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+\frac{2}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt.
\end{aligned} \tag{18}$$

Eventually we obtain:

$$\begin{aligned}
& E\int_0^T e^{\beta A(s)}|Y(s)|^2ds+E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt \\
\leq\ & 11E\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+9E\int_0^T|\psi(t)|^2dt\left(\int_t^T e^{\beta A(u)}d\beta A(u)\right) \\
& +11E\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2dt+9E\int_0^T\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2d\beta A(s)dt.
\end{aligned}$$

Furthermore, it follows that:

$$\begin{aligned}
& E\int_0^T e^{\beta A(t)}|Y(t)|^2dt+E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt \\
\leq\ & 20Ee^{\beta A(T)}\int_0^T|\psi(t)|^2dt+\frac{47}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt.
\end{aligned}$$

□

Remark 3.1 *If we define $Z(t,s),(0\le s<t\le T)$ by the following relation*

$$Y(t)=E^{\mathcal{F}_S}Y(t)+\int_S^t Z(t,s)dW(s),\quad t\in[S,T],\quad \forall S\in[0,T].$$

Then BSVIE (4) admits a unique M-solution in $\mathcal{H}^{2,\beta}[0,T]$.

3.2 The Lipschitz case

In this subsection, we give the existence and uniqueness of M-solution under Lipschitz condition with a much more convenient method. We require $r_i(s)$ to be deterministic functions $(i=1,2,3)$. First we give a theorem when $L(t,s)$ is bounded.

Theorem 3.1 *Let (H1) hold, $\psi(\cdot)\in L^{2,\beta}_{\mathcal{F}_T}[0,T]$ and $r_i(s)$ $(i=1,2,3)$ are deterministic functions, $L(t,s)$ is bounded, then (2) admits a unique M-solution in $\mathcal{H}^{2,\beta}[0,T]$.*

Proof. When $A(\cdot)$ is deterministic function, by the definition of M-solution we see that

$$\begin{aligned}
& E\int_0^T\int_0^t e^{\beta A(s)}|Z(t,s)|^2dsdt\\
\le\ & E\int_0^T e^{\beta A(t)}\int_0^t|Z(t,s)|^2dsdt\\
=\ & \int_0^T e^{\beta A(t)}E\int_0^t|Z(t,s)|^2dsdt\\
\le\ & E\int_0^T e^{\beta A(t)}|Y(t)|^2dt. \qquad (19)
\end{aligned}$$

Then

$$\begin{aligned}
& E\int_0^T e^{\beta A(t)}|Y(t)|^2dt+E\int_0^T\int_0^T e^{\beta A(s)}|Z(t,s)|^2dsdt\\
\le\ & 2E\int_0^T e^{\beta A(t)}|Y(t)|^2dt+E\int_0^T\int_t^T e^{\beta A(s)}|Z(t,s)|^2dsdt\\
\le\ & CE\int_0^T e^{\beta A(t)}|\psi(t)|^2dt+CE\int_0^T e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2dt\\
& +CE\int_0^T|\psi(t)|^2\left(\int_t^T e^{\beta A(u)}d\beta A(u)\right)dt\\
& +CE\int_0^T\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2d\beta A(s)dt\\
\le\ & CEe^{\beta A(T)}\int_0^T|\psi(t)|^2dt+\frac{C}{\beta}E\int_0^T\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt. \qquad (20)
\end{aligned}$$

Let $\mathcal{M}^{2,\beta}[0,T]$ be the space of all $(y(\cdot),z(\cdot,\cdot))\in\mathcal{H}^{2,\beta}[0,T]$ such that

$$y(t)=Ey(t)+\int_0^t z(t,s)dW(s),\quad t\in[0,T].$$

Clearly, it is a nonempty closed subspace of $\mathcal{H}^{2,\beta}[0,T]$. Now we consider the following BSVIE:

$$Y(t) = \psi(t) + \int_t^T g(t,s,y(s),z(t,s),z(s,t))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \quad (21)$$

for any $\psi(\cdot) \in L_T^{2,\beta}[0,T]$ and $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^{2,\beta}[0,T]$. Hence by Remark 3.1 we know that (21) admits a unique M-solution $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{M}^{2,\beta}[0,T]$, and we can define a map $\Theta : \mathcal{M}^{2,\beta}[0,T] \to \mathcal{M}^{2,\beta}[0,T]$ by

$$\Theta(y(\cdot), z(\cdot,\cdot)) = (Y(\cdot), Z(\cdot,\cdot)), \quad \forall (y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^{2,\beta}[0,T].$$

Let $(\overline{y}(\cdot), \overline{z}(\cdot,\cdot)) \in \mathcal{M}^{2,\beta}[0,T]$ and $\Theta(\overline{y}(\cdot), \overline{z}(\cdot,\cdot)) = (\overline{Y}(\cdot), \overline{Z}(\cdot,\cdot))$. From (20) we deduce that,

$$\begin{aligned}
& E\int_0^T e^{\beta A(t)} |Y(t) - \overline{Y}(t)|^2 dt + E\int_0^T \int_0^T e^{\beta A(s)} |Z(t,s) - \overline{Z}(t,s)|^2 ds dt \\
\leq \; & CE\int_0^T e^{\beta A(t)} \left| \int_t^T H(t,s)ds \right|^2 dt + CE\int_0^T \int_t^T e^{\beta A(s)} \left| \int_s^T H(t,u)du \right|^2 d\beta A(s)dt \\
\leq \; & CE\int_0^T e^{\beta A(t)} \left| \int_t^T G(t,s)ds \right|^2 dt + CE\int_0^T \int_t^T e^{\beta A(s)} \left| \int_s^T G(t,u)du \right|^2 d\beta A(s)dt \\
\leq \; & \frac{C}{\beta} E\int_0^T \int_t^T e^{\beta A(s)} L^2(t,s) |y(s) - \overline{y}(s)|^2 ds dt \\
& + \frac{C}{\beta} E\int_0^T \int_t^T e^{\beta A(s)} L^2(t,s) |z(t,s) - \overline{z}(t,s)|^2 ds dt \\
& + \frac{C}{\beta} E\int_0^T \int_t^T e^{\beta A(s)} L^2(t,s) |z(s,t) - \overline{z}(s,t)|^2 ds dt \\
\leq \; & \frac{C}{\beta} E\int_0^T e^{\beta A(s)} |y(s) - \overline{y}(s)|^2 ds + \frac{C}{\beta} E\int_0^T \int_t^T e^{\beta A(s)} |z(t,s) - \overline{z}(t,s)|^2 ds dt,
\end{aligned}$$

where

$$\begin{aligned}
H(t,s) &= g(t,s,y(s),z(t,s),z(s,t)) - g(t,s,\overline{y}(s),\overline{z}(t,s),\overline{z}(s,t)), \\
G(t,s) &= L(t,s)(r_1(s)|y(s) - \overline{y}(s)| \\
& \quad + r_2(s)|z(t,s) - \overline{z}(t,s)| + r_3(s)|z(s,t) - \overline{z}(s,t)|).
\end{aligned}$$

The fourth inequality above holds from that:

$$\begin{aligned}
& E\int_0^T \int_t^T e^{\beta A(s)} |z(s,t) - \overline{z}(s,t)|^2 ds dt \\
= \; & E\int_0^T e^{\beta A(t)} \int_0^t |z(t,s) - \overline{z}(t,s)|^2 ds dt \leq E\int_0^T e^{\beta A(t)} |y(t) - \overline{y}(t)|^2 dt. \quad (22)
\end{aligned}$$

Choosing a sufficient large number β so that $\frac{C}{\beta} < 1$, then the mapping Θ is contracted from $\mathcal{H}^{2,\beta}[0,T]$ onto itself. Thus there exists a unique fixed point which is the unique M-solution of BSVIE (2). $\square$

Remark 3.2 *From expression (19) and (22), we can know the reason for assuming $r_i(s)$ to be deterministic. When $A(s)$ is bounded (or continuous and deterministic) the norm of $\mathcal{H}^{2,\beta}[0,T]$ is equivalent to the norm of $\mathcal{H}^2[0,T]$, then BSVIE (2) admits a unique M-solution in $\mathcal{H}^2[0,T]$. As a result, we can finish the proof of uniqueness and existence of M-solution in $\mathcal{H}^2[0,T]$ with only one step, which is much more convenient than the four steps in [17].*

In Theorem 3.1 the assumption on the coefficient $L(t,s)$ is stronger than the one in [17]. Next we will see that we can relax it, but we need introduce a new non-negative process $A^*(t)$ and another assumption on $\alpha^2(t)$. We denote $A^*(t) = \int_0^t \alpha^{\frac{2p}{2-p}}(s)ds$ $(1 < p < 2)$, furthermore, we assume that $\alpha^2(s) \geq 1, t \in [0,T]$. Obviously we have $A^*(t) \geq A(t), t \in [0,T]$. We will obtain some estimates which are important in the proof of the following theorem. For arbitrary $1 < p < 2$, we obtain

$$
\begin{aligned}
& \left(\int_r^T |f(t,s)|^p ds\right)^{\frac{2}{p}} \\
= & \left(\int_r^T e^{-\tau A^*(s)}\alpha^p(s)e^{\tau A^*(s)}\frac{|f(t,s)|^p}{\alpha^p(s)}ds\right)^{\frac{2}{p}} \\
\leq & \left(\int_r^T e^{-\tau A^*(s)\frac{2}{2-p}}\alpha^{\frac{2p}{2-p}}(s)ds\right)^{\frac{2-p}{p}}\left(\int_r^T e^{\tau A^*(s)\frac{2}{p}}\frac{|f(t,s)|^2}{\alpha^2(s)}ds\right) \\
\leq & \left(\frac{2-p}{2\tau}\right)^{\frac{2-p}{p}} e^{-\tau A^*(r)\frac{2}{p}}\int_r^T e^{\tau A^*(s)\frac{2}{p}}\frac{|f(t,s)|^2}{\alpha^2(s)}ds, \quad t,r \in [0,T],
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
\left|\int_s^T f(t,u)du\right|^2 = & \left|\int_s^T e^{\frac{-rA^*(u)}{2}}\alpha(u)e^{\frac{rA^*(u)}{2}}\frac{|f(t,u)|}{\alpha(u)}du\right|^2 \\
\leq & \int_s^T e^{-rA^*(u)}\alpha^2(u)du \cdot \int_s^T e^{rA^*(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du \\
\leq & \int_s^T e^{-rA^*(u)}\alpha^{\frac{2p}{2-p}}(u)du \cdot \int_s^T e^{rA^*(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du \\
\leq & \frac{1}{r}e^{-rA^*(s)}\int_s^T e^{rA^*(u)}\frac{|f(t,u)|^2}{\alpha^2(u)}du, \quad t,s \in [0,T].
\end{aligned} \tag{24}
$$

In (23), let $\tau = \frac{p}{2}\beta$, we arrive at

$$
\begin{aligned}
& \left(\int_r^T |f(t,s)|^p ds\right)^{\frac{2}{p}} \\
\leq & \left(\frac{2-p}{p}\right)^{\frac{2-p}{p}}\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} e^{-\beta A^*(r)}\int_r^T e^{\beta A^*(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}ds.
\end{aligned} \tag{25}
$$

Then we have

Theorem 3.2 *Let (H1) hold, $\psi(\cdot) \in L^{2,\beta}_{\mathcal{F}_T}[0,T]$, $\alpha^2(s) \geq 1$, and $L(t,s)$ satisfies:*

$$\sup_{t\in[0,T]} \left(\int_t^T L^q(t,s)ds \right)^{\frac{2}{q}} < \infty, \tag{26}$$

where q is a constant and $q > 2$, $r_i(s)$ are determined functions, $A^*(s)$ is bounded (or continuous), then BSVIE (2) admits a unique M-solution in $\mathcal{H}^2[0,T]$.

Proof. Thanks to (24), we can replace $A(s)$ in Lemma 3.1 with $A^*(s)$ and then (20) becomes:

$$\begin{aligned}
& E\int_0^T e^{\beta A^*(t)}|Y(t)|^2dt + E\int_0^T\int_0^T e^{\beta A^*(s)}|Z(t,s)|^2dsdt \\
\leq\ & 2E\int_0^T e^{\beta A^*(t)}|Y(t)|^2dt + E\int_0^T\int_t^T e^{\beta A^*(s)}|Z(t,s)|^2dsdt \\
\leq\ & CE\int_0^T e^{\beta A^*(t)}|\psi(t)|^2dt + CE\int_0^T e^{\beta A^*(t)}\left|\int_t^T f(t,s)ds\right|^2 dt \\
& +CE\int_0^T |\psi(t)|^2 \left(\int_t^T e^{\beta A^*(u)}d\beta A^*(u)\right)dt \\
& + CE\int_0^T\int_t^T e^{\beta A^*(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A^*(s)dt \\
\leq\ & CEe^{\beta A^*(T)}\int_0^T |\psi(t)|^2dt + \frac{C}{\beta}E\int_0^T\int_t^T e^{\beta A^*(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}dsdt.
\end{aligned} \tag{27}$$

The former part is the same as the corresponding part in Theorem 3.1, thus we only state the rest. Note that $Y(\cdot), \overline{Y}(\cdot), Z(\cdot,\cdot), \overline{Z}(\cdot,\cdot)$ have the same meaning as above. It follows that,

$$\begin{aligned}
& E\int_0^T e^{\beta A^*(t)}|Y(t)-\overline{Y}(t)|^2dt + E\int_0^T\int_0^T e^{\beta A^*(s)}|Z(t,s)-\overline{Z}(t,s)|^2dsdt \\
\leq\ & CE\int_0^T e^{\beta A^*(t)}\left|\int_t^T H(t,s)ds\right|^2 dt + CE\int_0^T\int_t^T e^{\beta A^*(s)}\left|\int_s^T H(t,u)du\right|^2 d\beta A^*(s)dt \\
\leq\ & CE\int_0^T e^{\beta A^*(t)}\left|\int_t^T G(t,s)ds\right|^2 dt + CE\int_0^T\int_t^T e^{\beta A^*(s)}\left|\int_s^T G(t,u)du\right|^2 d\beta A^*(s)dt \\
\leq\ & CE\int_0^T e^{\beta A^*(t)}\left(\int_t^T L^q(t,s)ds\right)^{\frac{2}{q}}\left(\int_t^T U^p(t,s)ds\right)^{\frac{2}{p}}dt \\
& +CE\int_0^T\int_t^T e^{\beta A^*(s)}\left(\int_s^T L^q(t,u)du\right)^{\frac{2}{q}}\left(\int_s^T U^p(t,u)du\right)^{\frac{2}{p}}d\beta A^*(s)dt \\
\leq\ & C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_0^T\int_t^T e^{\beta A^*(s)}|y(s)-\overline{y}(s)|^2dsdt
\end{aligned}$$

$$
\begin{aligned}
&+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_0^T\int_t^T e^{\beta A^*(s)}|z(t,s)-\overline{z}(t,s)|^2dsdt \\
&+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_0^T\int_t^T e^{\beta A^*(s)}|z(s,t)-\overline{z}(s,t)|^2dsdt \\
\leq\ & C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_0^T e^{\beta A^*(s)}|y(s)-\overline{y}(s)|^2ds \\
&+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_0^T\int_t^T e^{\beta A^*(s)}|z(t,s)-\overline{z}(t,s)|^2dsdt,
\end{aligned}
$$

where

$$
\begin{aligned}
H(t,s) &= g(t,s,y(s),z(t,s),z(s,t))-g(t,s,\overline{y}(s),\overline{z}(t,s),\overline{z}(s,t)),\\
G(t,s) &= L(t,s)(r_1(s)|y(s)-\overline{y}(s)| \\
&\quad +r_2(s)|z(t,s)-\overline{z}(t,s)|+r_3(s)|z(s,t)-\overline{z}(s,t)|),\\
U(t,s) &= r_1(s)|y(s)-\overline{y}(s)| \\
&\quad +r_2(s)|z(t,s)-\overline{z}(t,s)|+r_3(s)|z(s,t)-\overline{z}(s,t)|.
\end{aligned}
$$

Choosing a sufficient large number β so that $C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}}<1$. Then the mapping Θ is contracted from $\mathcal{H}^{2,\beta}[0,T]$ onto itself. Because of the assumption on $A^*(s)$, we know that BSVIE (2) admits a unique M-solution in $\mathcal{H}^2[0,T]$. $\square$

Remark 3.3 *In [17], the assumption on $L(t,s)$ is $\sup_{t\in[0,T]}\int_t^T L^{2+\epsilon}(t,s)ds<\infty$, where ϵ is a positive constant. When $r_i(s)$ $(i=1,2,3)$ are constants, then from (26) we know that the assumptions on Lipschitz coefficients are the same as the one in [17]. On the other hand, here we assume that $A(t)$ is bounded, so if $\psi(\cdot)\in L^2_{\mathcal{F}_T}[0,T]$ and $E\int_0^T\int_t^T g_0(t,s)dsdt<\infty$, we can get $\psi(\cdot)\in L^{2,\beta}_{\mathcal{F}_T}[0,T]$ and $E\int_0^T\int_t^T e^{A(s)}g_0(t,s)dsdt<\infty$, then by Theorem 3.2 we can also get the existence and uniqueness of M-solution in $\mathcal{H}^2[0,T]$ which is one of the main results in [17].*

Remark 3.4 *We can use same argument as above to give the stability estimate, for any $S\in[0,T]$,*

$$
\begin{aligned}
& E\int_S^T|Y(t)-\overline{Y}(t)|^2dt+\int_S^T\int_S^T|Z(t,s)-\overline{Z}(t,s)|^2dsdt \\
\leq\ & CE\int_S^T|\psi(t)-\overline{\psi}(t)|^2dt+CE\int_S^T\left(\int_t^T|g-\overline{g}|ds\right)^2dt, \qquad (28)
\end{aligned}
$$

where $g=g(t,s,Y(s),Z(t,s),Z(s,t))$ and $\overline{g}=\overline{g}(t,s,Y(s),Z(t,s),Z(s,t))$, $\overline{\psi}(\cdot)\in L^2_{\mathcal{F}_T}[0,T]$ and $(\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot))\in\mathcal{H}^2[0,T]$ be the adapted M-solution of (2) with g and $\psi(\cdot)$ replaced by $\overline{g}$ and $\overline{\psi}(\cdot)$, respectively.

Similarly we can obtain the existence and uniqueness of the adapted solution for BSVIE (3).

Theorem 3.3 *Let (H1) hold, $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, $\alpha^2(s) \geq 1$, and $L(t,s)$ satisfies:*

$$\sup_{t\in[0,T]} \left(\int_t^T L^q(t,s)ds \right)^{\frac{2}{q}} < \infty,$$

where q is a constant and $q > 2$, $r_i(s)$ are two adapted processed, $A^*(s)$ is bounded, then BSVIE (3) admits a unique adapted solution in $\mathcal{H}^2[0,T]$.

Proof. We can get the result by Lemma 3.1 and the fixed point theorem, so we omit it. $\square$

Remark 3.5 *Here we let the Lipschitz coefficient be stochastic because we do not need to consider the value of $Z(t,s)$ $(0 \leq s \leq t \leq T)$ for the adapted solution of BSVIE (3).*

3.3 The non-Lipschitz case

In this subsection, we will consider the unique existence of adapted solution of BSVIE (3) and M-solution of BSVIE (2) under non-Lipschitz condition. We assume that

(H2) For all y, $\overline{y} \in R^m$, z, $\overline{z}$, ζ, $\overline{\zeta} \in R^{m\times d}$, and $(t,s) \in \Delta^c$

$$\begin{aligned} & |g(t,s,y,z,\zeta) - g(t,s,\overline{y},\overline{z},\overline{\zeta})| \\ \leq \ & L(t,s)(r_1(s)(\rho(|y-\overline{y}|^2))^{\frac{1}{2}} + r_2(s)|z-\overline{z}| + r_3(s)|\zeta - \overline{\zeta}|), \end{aligned}$$

where ρ is an increasing concave function from R_+ to R_+ such that $\rho(0) = 0$, and $\int_{0+} \frac{du}{\rho(u)} = \infty$. $L(t,s)$ is a deterministic non-negative function.

Since ρ is concave and $\rho(0) = 0$, one can find a pair of positive constants a and b such that $\rho(u) \leq a + bu$, for all $u \geq 0$. Next we use the argument in Lemma 3.1 to give another estimate which plays a critical role in the next. The following form of BSVIE

$$Y(t) = \xi + \int_t^T g(t,s,Y(s),Z(t,s))ds - \int_t^T Z(t,s)dW(s), \tag{29}$$

under non-Lipschitz coefficient was considered in [14]. The author also gave a critical estimate by using the Itô formula to $e^{\beta t}|Y(t)|^2$ to give an estimate for $e^{\beta t}|Y(t)|^2 + E^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z(t,s)|^2 ds$ where $(Y(\cdot), Z(\cdot,\cdot))$ is the adapted solution of equation (29). Now we give another estimate for $Ee^{\beta t}|Y(t)|^2 + E\int_t^T e^{\beta s}|Z(t,s)|^2 ds$ by the same method in Lemma 3.1 without involving Itô formula. We have:

Lemma 3.2 *Let $\psi(\cdot) \in L^{2,\beta}_{\mathcal{F}_T}[0,T]$, the assumptions on f is the same as in Lemma 3.1, $(Y(\cdot), Z(\cdot,\cdot))$ is the adapted solution of (4), then for almost every $t \in [0,T]$, we have the following estimate:*

$$\begin{aligned} & Ee^{\beta A(t)}|Y(t)|^2 + E\int_t^T e^{\beta A(s)}|Z(t,s)|^2 ds \\ \leq \ & Ee^{\beta A(T)}|\psi(t)|^2 + \frac{C}{\beta}E\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}ds. \end{aligned}$$

Proof. It follows from (7) that

$$Y(t) \;=\; \lambda(t,t) = E^{\mathcal{F}_t}\left\{\psi(t) + \int_t^T f(t,s)ds\right\}.$$

Then

$$e^{\beta A(t)}|Y(t)|^2 \;\le\; 2e^{\beta A(t)}E(|\psi(t)|^2\big|\,\mathcal{F}_t) + 2e^{\beta A(t)}E\left(\left|\int_t^T f(t,s)ds\right|^2\middle|\,\mathcal{F}_t\right),$$

thus

$$Ee^{\beta A(t)}|Y(t)|^2 \;\le\; 2E(e^{\beta A(t)}|\psi(t)|^2) + 2E\left(e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2\right).$$

By taking $r = t$ in (11), we claim that

$$\begin{aligned}
&E\int_t^T e^{\beta A(s)}|Z(t,s)|^2ds \\
=\;& E\int_t^T e^{\beta A(s)}\left(\int_s^T |Z(t,u)|^2du\right)d\beta A(s) + Ee^{\beta A(t)}\int_t^T |Z(t,u)|^2du.
\end{aligned}$$

In the light of the proof in Lemma 3.1, it can be easily checked that,

$$\begin{aligned}
&E\int_t^T e^{\beta A(s)}|Z(t,s)|^2ds \\
\le\;& CEe^{\beta A(T)}|\psi(t)|^2 + CE\left(e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2\right) \\
&+CE\left(\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s)\right).
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
&Ee^{\beta A(t)}|Y(t)|^2 + E\int_t^T e^{\beta A(s)}|Z(t,s)|^2ds \\
\le\;& CEe^{\beta A(T)}|\psi(t)|^2 + CE\left(e^{\beta A(t)}\left|\int_t^T f(t,s)ds\right|^2\right) \\
&+CE\left(\int_t^T e^{\beta A(s)}\left|\int_s^T f(t,u)du\right|^2 d\beta A(s)\right) \\
\le\;& CEe^{\beta A(T)}|\psi(t)|^2 + \frac{C}{\beta}E\int_t^T e^{\beta A(s)}\frac{|f(t,s)|^2}{\alpha^2(s)}ds.
\end{aligned}$$

The conclusion thus follows. $\square$

Remark 3.6 *When $r_i(s)(i=1,2)$ are constants, the above estimate becomes:*

$$\begin{aligned} & Ee^{\beta t}|Y(t)|^2 + E\int_t^T e^{\beta s}|Z(t,s)|^2 ds \\ \le\ & CEe^{\beta T}|\psi(t)|^2 + \frac{C}{\beta}E\int_t^T e^{\beta s}|f(t,s)|^2 ds \end{aligned} \tag{30}$$

which is similar to the one in [14]:

$$\begin{aligned} & e^{\beta t}|Y(t)|^2 + E^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z(t,s)|^2 ds \\ \le\ & e^{\beta T}E^{\mathcal{F}_t}|\xi|^2 + \frac{1}{\beta}E\int_t^T e^{\beta s}|f(t,s)|^2 ds. \end{aligned} \tag{31}$$

Though the estimate (31) is stronger than (30), (30) still can guarantee that equation (3) admits a unique adapted solution under non-Lipschitz coefficients.

We have

Theorem 3.4 *Let (H2) hold, g is independent of $Z(s,t)$, $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, $r_i(s)$ are adapted processes, $A(t)$ is bounded, $L(t,s)$ satisfies:*

$$\sup_{t\in[0,T]}\left(\int_t^T L^q(t,s)ds\right)^{\frac{2}{q}} < \infty,$$

where $q>2$ is a constant, then (?) admits a unique adapted solution in $\mathcal{H}_t^2[0,T]$.

Proof. The proof can be obtained by combining the estimate in Lemma 3.1 and the proof in [14], so we omit it. □

When $r_i(s)$ is a constant and $\psi(\cdot)=\xi$, g is independent of $Z(s,t)$, then we get the result in [14]:

Corollary 3.1 *Let (H2) hold, $r_i(s)=1$, $L(t,s)=k$, k is a constant, then (3) admits a unique adapted solution in $\mathcal{H}_t^2[0,T]$.*

Obviously we can also get the unique existence of M-solution of (3) as above. However, as to the general form of (2), there is an expression $E\int_t^T|Z(s,t)|^2ds$ which is hard to estimate directly, see [17], so we have to adopt new method to deal with it. We need to prepare some results in order to be able to derive this claim.

Lemma 3.3 *For any $t\in[0,T]$, $f(s):[t,T]\to R^+$, $c(x):R\to R$ is a concave function, $\int_0^T f(s)ds<\infty$. Then we have*

$$\frac{1}{T-t}\int_t^T c(f(s))ds \le c\left(\frac{1}{T-t}\int_t^T f(s)ds\right).$$

Proof. Obviously $-c(x)$ is a convex function, for fixed $x \in R$, $\forall\ y_1 > x$, $y_2 < x$, we have (see [13]).

$$\frac{-c(y_1)+c(x)}{y_1 - x} \geq -c_+^{'}(x) \geq -c_-^{'}(x) \geq \frac{-c(y_2)+c(x)}{y_2 - x},$$

thus there exists a $k \in [-c_-^{'}(x), -c_+^{'}(x)]$ so that $\forall y \in R$, $-c(y) \geq -c(x) + k \cdot (y-x)$, i.e., $c(y) \leq c(x) - k(y-x)$. For any fixed $t \in [0,T]$, $s \in [t,T]$,

$$x = \frac{1}{T-t}\int_t^T f(s)ds,\ y = f(s),$$

then

$$c(f(s)) \leq c\left(\frac{1}{T-t}\int_t^T f(s)ds\right) - k \cdot \left(f(s) - \frac{1}{T-t}\int_t^T f(s)ds\right),$$

thus we get the conclusion above. $\square$

We are now ready to establish the last result of this paper.

Theorem 3.5 *Let (H2) hold, $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, $r_i(s)$ are deterministic function, $A(t)$ is bounded, $L(t,s)$ satisfies:*

$$\sup_{t\in[0,T]} \left(\int_t^T L^q(t,s)ds\right)^{\frac{2}{q}} < \infty,$$

where $q > 2$ is a constant, then (2) admits a unique M-solution in $\mathcal{H}^2[0,T]$.

Proof. Uniqueness: Let $(Y_i, Z_i) \in \mathcal{H}^2[0,T]$ $(i = 1,2)$ be any two M-solutions. By defining

$$\widehat{Y}(t) = Y_1(t) - Y_2(t);\ \widehat{Z}(t,s) = Z_1(t,s) - Z_2(t,s),\ t,s \in [0,T],$$

we arrive that

$$\begin{aligned} &\widehat{Y}(t) + \int_t^T \widehat{Z}(t,s)dW(s) \\ = &\int_t^T [g(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)) - g(t,s,Y_2(s),Z_2(t,s),Z_2(s,t))]ds. \end{aligned}$$

Note that $\widehat{Y}(T) = 0$. For arbitrary $u \in [0,T)$, we obtain the following results in the same way as Theorem 3.2,

$$\begin{aligned} &E\int_u^T e^{\beta A^*(t)}|\widehat{Y}(t)|^2dt + E\int_u^T\int_t^T e^{\beta A^*(s)}|\widehat{Z}(t,s)|^2dsdt \\ \leq\ & C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_u^T\int_t^T e^{\beta A^*(s)}\rho(|\widehat{Y}(s)|^2)dsdt \\ &+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_u^T\int_t^T e^{\beta A^*(s)}|\widehat{Z}(t,s)|^2dsdt + C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}} E\int_u^T e^{\beta A^*(t)}|\widehat{Y}(t)|^2dt. \end{aligned}$$

By choosing a suitable β, we deduce the following

$$E\int_u^T e^{\beta A^*(t)}|\widehat{Y}(t)|^2dt \leq CE\int_u^T\int_t^T e^{\beta A^*(s)}\rho(|\widehat{Y}(s)|^2)dsdt,$$

consequently,

$$\begin{aligned}\frac{1}{T-u}E\int_u^T|\widehat{Y}(t)|^2dt &\leq CE\int_u^T\frac{1}{T-t}\int_t^T\rho(|\widehat{Y}(s)|^2)dsdt\\ &\leq C\int_u^T\rho\left(\frac{1}{T-t}\int_t^T E|\widehat{Y}(s)|^2ds\right)dt.\end{aligned}$$

Due to Bihari's inequality (see [2]) we get that $\frac{1}{T-u}E\int_u^T|\widehat{Y}(t)|^2dt=0$, $u\in[0,T)$, thus $\widehat{Y}(t)=0$ as well as $\widehat{Z}(t,s)=0$, $t,s\in[0,T]$. a.e.

Existence: Let $Y_0(t)\equiv 0$, and define recursively (Y_n,Z_n) by the following equations with the help of Theorem 3.2

$$Y_n(t)=\psi(t)+\int_t^T g(t,s,Y_{n-1}(s),Z_n(t,s),Z_n(s,t))ds-\int_t^T Z_n(t,s)dW(s). \tag{32}$$

By setting

$$\widehat{Y}_{n,k}(t)=Y_n(t)-Y_k(t);\ \widehat{Z}_{n,k}(t,s)=Z_n(t,s)-Z_k(t,s),\ t,s\in[0,T],$$

and choosing a suitable β, we claim that

$$\begin{aligned}&E\int_u^T e^{\beta A^*(t)}|\widehat{Y}_{n,k}(t)|^2dt+E\int_u^T\int_t^T e^{\beta A^*(s)}|\widehat{Z}_{n,k}(t,s)|^2dsdt\\ \leq\ & CE\int_u^T\int_t^T e^{\beta A^*(s)}\rho(|\widehat{Y}_{n-1,k-1}(s)|^2)dsdt,\end{aligned}$$

then

$$\frac{1}{T-u}E\int_u^T|\widehat{Y}_{n,k}(t)|^2dt\leq C\int_u^T\rho\left(\frac{1}{T-t}\int_t^T E|\widehat{Y}_{n-1,k-1}(s)|^2ds\right)dt.$$

Set $Q(u)=\limsup_{n,k\to\infty}E\int_u^T|\widehat{Y}_{n,k}(t)|^2dt$, it is easy to show that $S(u)=\sup_{n\geq 0}E\int_u^T|Y_n(t)|^2dt$ is bounded. In fact, using the similar trick as in Theorem 3.2, we obtain that

$$\begin{aligned}&E\int_u^T e^{\beta A^*(t)}|Y_n(t)|^2dt+E\int_u^T\int_t^T e^{\beta A^*(s)}|Z_n(t,s)|^2dsdt\\ \leq\ & CE\int_u^T e^{\beta A^*(t)}|\psi(t)|^2dt+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}}E\int_u^T\int_t^T e^{\beta A^*(s)}|g_0(t,s)|^2dsdt\\ &+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}}E\int_u^T\int_t^T e^{\beta A^*(s)}(a+b|Y_{n-1}(s)|^2)dsdt\\ &+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}}E\int_u^T\int_t^T e^{\beta A^*(s)}|Z_n(t,s)|^2dsdt+C\left(\frac{1}{\beta}\right)^{\frac{2-p}{p}}E\int_u^T e^{\beta A^*(t)}|Y_n(t)|^2dt,\end{aligned}$$

thus by choosing β we have

$$\begin{aligned} E\int_u^T |Y_n(t)|^2dt &\leq C + CE\int_u^T |\psi(t)|^2dt \\ &\quad + CE\int_u^T\int_t^T |g_0(t,s)|^2dsdt + E\int_u^T\int_t^T |Y_{n-1}(s)|^2dsdt. \end{aligned}$$

In view of Gronwall's inequality we obtain that $S(u)$is bounded. Then by Fatou's lemma, Bihari's inequality and noting that ρ is increasing, we deduce that for almost $u \in [0,T]$, $Q(u) = 0$, and it follows that

$$\lim_{n,k\to\infty} E\int_0^T |Y_n(t) - Y_k(t)|^2dt = 0,$$

hence there is a Y such that

$$\lim_{n\to\infty} E\int_0^T |Y_n(t) - Y(t)|^2dt = 0.$$

Similarly there is a Z such that

$$\begin{aligned} \lim_{n\to\infty} E\int_0^T\int_t^T |Z_n(t,s) - Z(t,s)|^2dsdt &= 0, \\ \lim_{n\to\infty} E\int_0^T\int_0^t |Z_n(t,s) - Z(t,s)|^2dsdt &\leq \lim_{n\to\infty} E\int_0^T |Y_n(t) - Y(t)|^2dt = 0. \end{aligned}$$

By taking the limits for BSVIE (32), one can finds that (Y,Z) is a M-solution of BSVIE (2). $\square$

At last we want to give a simple example to show the unique existence of adapted solution (or M-solution) under non-Lipschitz condition. As shown in [8] or [9], the following two functions satisfy the assumption of ρ in (H2),

$$\rho_1(x) = \begin{cases} x\ln(x^{-1}), & x \in [0,\delta], \\ \delta\ln(\delta^{-1}) + \dot{\rho}_1(\delta-)(x-\delta), & x > \delta, \end{cases}$$

$$\rho_2(x) = \begin{cases} x\ln(x^{-1})\ln\ln(x^{-1}), & x \in [0,\delta], \\ \delta\ln(\delta^{-1})\ln\ln(\delta^{-1}) + \dot{\rho}_2(\delta-)(x-\delta), & x > \delta, \end{cases}$$

with $\delta \in (0,1)$ being sufficiently small. However, the explicit form of ρ_i is not easy to get, so now we will give another example to avoid this problem.

Let us consider the following equation

$$Y(t) = \psi(t) + \int_t^T L(t,s)[f(|Y(s)|) + |Z(t,s)| + |Z(s,t)|]ds - \int_t^T Z(t,s)dW(s), \quad (33)$$

where $f : R \to [0,\infty)$ is defined by

$$f(x) = \begin{cases} 0 & x = 0, \\ |x|\left[\ln(1+|x|^{-1})\right]^{\frac{1}{2}} & 0 < |x| < \delta, \\ \delta\left[\ln(1+|\delta|^{-1})\right]^{\frac{1}{2}} & |x| \geq \delta, \end{cases}$$

$L(t,s)$ satisfies $\sup_{t\in[0,T]} \int_t^T L^q(t,s)ds < \infty$, where $q > 2$ is a constant. It can be shown that $|f(y) - f(\overline{y})| \leq \rho(|y-\overline{y}|^2)^{\frac{1}{2}}$ where ρ can be defined by

$$\rho(x) = \begin{cases} 0 & x = 0, \\ x\ln(1+x^{-1}) & 0 < x < 1, \\ \ln 2 & x \geq 1, \end{cases}$$

We refer the reader to [3] for the proof. Then by Theorem 3.5, we deduce that BSVIE (33) admits a unique M-solution. Note that we can give the example for adapted solution in a similar way.

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